Theorem: When $\omega_1 > 0$, there exist an optimal defender strategy that is a pure strategy.

Proof:

R_i^d, P_i^d, R_i^a, P_i^a are the reward (R) and penalty (P) for the defender (superscript d) and the attacker (superscript a) respectively for target i. Let $x =< x_i >$ be a defender strategy, and x_i is the coverage probability of target i. Let q_i be the attacking probability for target i. According to SUQR model,

$$q_i = \frac{e^{\omega_1 x_i + \omega_2 R_i^a + \omega_3 P_i^a}}{\sum_j e^{\omega_1 x_j + \omega_2 R_j^a + \omega_3 P_j^a}}$$

Let U_i^d be the defender's expected utility when target i is attacked, i.e., $U_i^d = (R_i^d - P_i^d)x_i + P_i^d$. Then defender's overall expected utility can be represented as

$$f(x) = \sum_i q_i U_i^d$$

Assume $\bar{x}$ is the optimal defender strategy. Let $\bar{S}$ be the set of targets with positive coverage probability, i.e., $\bar{S} = \{i|\bar{x}_i > 0\}$. Then $\forall i \in \bar{S}$, $\frac{\partial f}{\partial x_i}|_{\bar{x}} \geq 0$. Otherwise, a defender strategy with a lower coverage probability on target i will achieve a higher defender expected utility than $\bar{x}$, contradict with the optimality. Formally, let $\Delta_i = (0, 0, ..., \delta, 0, 0)$ be a vector with an infinitesimal positive value in i^{th} row. If $\frac{\partial f}{\partial x_i} < 0$, then $f(\bar{x} - \Delta_i) = f(\bar{x}) - \delta \frac{\partial f}{\partial x_i}|_{\bar{x}} > f(\bar{x})$.

Further, the targets in $\bar{S}$ can be devided into two subsets $\bar{S_1}$ and $\bar{S_2}$ where $\bar{S_1} = \{i|\bar{x_i} = 1\}$ and $\bar{S_2} = \{i|0 < \bar{x_i} < 1\}$. Then $\forall i, j \in \bar{S_2}$, $\frac{\partial f}{\partial x_i}|_{\bar{x}} = \frac{\partial f}{\partial x_j}|_{\bar{x}} \geq 0$. Otherwise, a defender strategy that moves a little bit coverage probability from a target with higher partial derivative to a target with a lower partial derivative will achieve a higher defender expected utlility than $\bar{x}$, contradict with the optimality. Formally, if $\frac{\partial f}{\partial x_i}|_{\bar{x}} > \frac{\partial f}{\partial x_j}|_{\bar{x}}$,

$$\begin{aligned}
f(\bar{x} + \Delta_i - \Delta_j) - f(\bar{x}) &= f(\bar{x} + \Delta_i - \Delta_j) - f(\bar{x} + \Delta_i) + f(\bar{x} + \Delta_i) - f(\bar{x}) && (1)\\
&= -\delta \frac{\partial f}{\partial x_j}|_{\bar{x}+\Delta_i} + \delta \frac{\partial f}{\partial x_i}|_{\bar{x}} && (2)\\
&= -\delta(\frac{\partial f}{\partial x_j}|_{\bar{x}} + \delta \frac{\partial f^2}{\partial x_i \partial x_j}|_{\bar{x}}) + \delta \frac{\partial f}{\partial x_i}|_{\bar{x}} && (3)\\
&= \delta(\frac{\partial f}{\partial x_i}|_{\bar{x}} - \frac{\partial f}{\partial x_j}|_{\bar{x}}) - \delta^2 \frac{\partial f^2}{\partial x_i \partial x_j}|_{\bar{x}} && (4)\\
&> 0 && (5)
\end{aligned}$$

The last inequality is achieved by neglecting the second order term. So $f(\bar{x} + \Delta_i - \Delta_j) > f(\bar{x})$.

Now we show that when $\omega_1 > 0$, moving a little a small coverage probability from one target in $\bar{S_2}$ to another will not decrease the defender's expected utilitly, i.e., $f(\bar{x} + \Delta_i - \Delta_j) - f(\bar{x}) \geq 0$ when $\omega_1 > 0$, $i, j \in \bar{S_2}$. Thus, we can always move the coverage probability on targets in $\bar{S_2}$ until some targets are coverd with probability 1 and others are covered with probability 0. Thus we get a new defender strategy $\tilde{x}$ with no less expected defender utility than $\bar{x}$ and the corresponding $\tilde{S_2} = \emptyset$. To prove this, select two targets $i, j \in \bar{S_2}$, as $\frac{\partial f}{\partial x_i}|_{\bar{x}} = \frac{\partial f}{\partial x_j}|_{\bar{x}}$, $f(\bar{x} + \Delta_i - \Delta_j) - f(\bar{x}) = -\delta^2 \frac{\partial f^2}{\partial x_i \partial x_j}|_{\bar{x}}$ according to line (4). As

$$\frac{\partial f}{\partial x_i}|_{\bar{x}} = \omega_1 q_i (U_i^d - f) + q_i (R_i^d - P_i^d) \geq 0$$

, we have

$$\omega_1 (f - U_i^d) \leq R_i^d - P_i^d \tag{6}$$

, and similarly,

$$\omega_1 (f - U_j^d) \leq R_j^d - P_j^d \tag{7}$$

.

So

$$\begin{aligned}
\frac{\partial f^2}{\partial x_i \partial x_j} &= \omega_1^2 q_i q_j (2f - U_i^d - U_j^d) - \omega_1 q_i q_j (R_i^d - P_i^d + R_j^d - P_j^d) && (8)\\
&= \omega_1 q_i q_j (\omega_1 (f - U_i^d) + \omega_1 (f - U_j^d) - (R_i^d - P_i^d + R_j^d - P_j^d)) && (9)\\
&\leq \omega_1 q_i q_j (R_i^d - P_i^d + R_j^d - P_j^d - (R_i^d - P_i^d + R_j^d - P_j^d)) && (10)\\
&= 0 && (11)
\end{aligned}$$

The inequality in line (10) comes from (6) and (7) and the fact the $\omega_1 > 0$. Thus we have $f(\bar{x} + \Delta_i - \Delta_j) - f(\bar{x}) = -\delta^2 \frac{\partial f^2}{\partial x_i \partial x_j}|_{\bar{x}} \geq 0$. So we can move the coverage probability between targets in $\bar{S_2}$ without a degredation in defender's expected utility. As a result, we get a new optiaml defender strategy $\tilde{x}$ whose coverage probabilities are chosen only from $0, 1$, i.e., a pure defender strategy.